1 Alternate definitions of a ridge

Below, we give two alternative, but similar, definitions of a ridge, the first being the more convenient and somewhat simpler definition of a *second-derivative ridge*, which was presented in the tutorial as the definition of a ridge. The second is a *curvature ridge*; key concepts in this geometric definition are that of principal curvatures and principal directions [**?**].

As an aside, curvature ridge definition given below is somewhat restrictive in the sense that it is assumed that the surface of interest represents the graph of a function and hence the ridge is defined in the domain of the graph, instead of on the surface itself. However the definition can easily be generalized for an arbitrary orientable surface.

Definition 1.1 *A **second-derivative ridge** of σ is an injective curve $\mathbf{c} : s \mapsto D$, where $s \in (a, b) \subset \mathbb{R}$, satisfying the following conditions for each s in the open interval (a, b):*

SR1. *The vectors $\mathbf{c}'(s)$ and $\nabla\sigma(\mathbf{c}(s))$ are parallel.*
SR2. $\Sigma(\mathbf{n}, \mathbf{n}) = \min_{\|\mathbf{u}\|=1} \Sigma(\mathbf{u}, \mathbf{u}) < 0$*, where $\mathbf{n}$ is a unit normal vector to the curve $\mathbf{c}(s)$ and Σ is thought of as a bilinear form evaluated at the point $\mathbf{c}(s)$.*

Definition 1.2 *Let $\mathcal{G} \subset \mathbb{R}^3$ denote the graph of $\sigma : D \subset \mathbb{R}^2 \mapsto \mathbb{R}$. Let $\pi : \mathcal{G} \to D$ be the standard projection map, with its associate tangent map $T\pi$. A **curvature ridge** of the graph $\mathcal{G}$ is an injective curve $\mathbf{c} : s \mapsto D$, where $s \in (a, b) \subset \mathbb{R}$, satisfying the following conditions for each s in the open interval (a, b):*

CR1. *The vectors $\mathbf{c}'(s) = \frac{d\mathbf{c}}{ds}$ and $\nabla\sigma(\mathbf{c}(s))$ are parallel .*
CR2. *Regard $\mathcal{G}$ as an orientated surface in $\mathbb{R}^3$. Let $\mathbf{p} = \mathbf{c}(s)$ and $\mathcal{G} \ni \tilde{\mathbf{p}} = \pi^{-1}(\mathbf{p})$. Let $k^u_{\tilde{\mathbf{p}}}$ and $k^l_{\tilde{\mathbf{p}}}$ denote the maximum and minimum principal curvatures of $\mathcal{G}$ at the point $\tilde{\mathbf{p}}$ with corresponding unit principal vectors $\tilde{\mathbf{u}}^u_{\tilde{\mathbf{p}}}$ and $\tilde{\mathbf{u}}^l_{\tilde{\mathbf{p}}}$. We require that $k^l_{\tilde{\mathbf{p}}} < 0$ and that $T\pi(\tilde{\mathbf{u}}^l_{\tilde{\mathbf{p}}})$ be normal to $\mathbf{c}'(s)$.*

The main difference between the two definitions is this: in **CR2** the curvature is measured with respect to the tangent plane to the graph of σ at each point, whereas in **SR2**, the curvature is always with respect to the xy-plane. That is, in the first definition, the downward direction is fixed and points toward the xy-plane [1], whereas in the second definition, the downward direction is always parallel to the normal vector field of the graph. The first definition is more intrinsic, whereas the second is more intuitive. As expected, one can prove the two measures are equal at local extremum points, at which the two planes are

[1] We assume $D \subset xy$-plane.

parallel. In the next section, we show that a second-derivative ridge is always a subset of a curvature ridge.

2 Equivalence Between Ridge Definitions

The relationships between the curvature measures used in the two previous definitions can be summarized as follows:

Theorem 2.1 *For each point $\mathbf{p} \in D$, let $\mathbf{t}$ be a vector of arbitrary length oriented along $\nabla\sigma$ and $\mathbf{n}$ be a vector of arbitrary length oriented orthogonal to $\mathbf{t}$ (if $\nabla\sigma = 0$, $\mathbf{t}$ can be arbitrarily oriented). Let $\gamma_n = \Sigma(\mathbf{n}, \mathbf{n})$ and $\gamma_t = \Sigma(\mathbf{t}, \mathbf{t})$. As before, let $\tilde{\mathbf{t}} = (T\pi)^{-1}\mathbf{t}$ and $\tilde{\mathbf{n}} = (T\pi)^{-1}\mathbf{n}$. Then we have the following relations:*

$$\gamma_n = \kappa\, k(\tilde{\mathbf{n}})$$
$$\gamma_t = \kappa^3\, k(\tilde{\mathbf{t}})$$

where $\kappa = \sqrt{1 + \left(\frac{\partial\sigma}{\partial x}\right)^2 + \left(\frac{\partial\sigma}{\partial y}\right)^2}$.

PROOF. Let the DLE field be given by the function $\sigma(x, y)$ and $\mathcal{G}$ denote the graph $z = \sigma(x, y)$. The unit normal field to $\mathcal{G}$ is given by

$$\mathbf{u} = \frac{1}{\kappa}\left(-\frac{\partial\sigma}{\partial x}, -\frac{\partial\sigma}{\partial y}, 1\right) . \tag{1}$$

By definition [?], the normal curvature in the direction $\tilde{\mathbf{n}}$ is given by

$$k(\tilde{\mathbf{n}}) = \tilde{\mathbf{n}} \cdot \nabla_{\tilde{\mathbf{n}}}\mathbf{u} \tag{2}$$

where $\nabla_{\tilde{\mathbf{n}}}\mathbf{u}$ is the covariant derivative of $\mathbf{u}$ with respect to $\tilde{\mathbf{n}}$.

Using this formula for an arbitrary vector $\mathbf{w} = (w_x, w_y, w_z)$, the curvature along $\mathbf{w}$ is given by

$$\begin{aligned} k(\mathbf{w}) = & \tfrac{1}{\kappa}\left(w_x^2 \tfrac{\partial^2\sigma}{\partial x^2} + 2w_x w_y \tfrac{\partial^2\sigma}{\partial x\partial y} + w_y^2 \tfrac{\partial^2\sigma}{\partial y^2}\right) \\ & - \tfrac{1}{\kappa^3}\left(\tfrac{\partial\sigma}{\partial x}\left(\tfrac{\partial\sigma}{\partial x}\tfrac{\partial^2\sigma}{\partial x\partial y} + \tfrac{\partial\sigma}{\partial y}\tfrac{\partial^2\sigma}{\partial y^2}\right) + \tfrac{\partial\sigma}{\partial y}\left(\tfrac{\partial\sigma}{\partial y}\tfrac{\partial^2\sigma}{\partial x\partial y} + \tfrac{\partial\sigma}{\partial x}\tfrac{\partial^2\sigma}{\partial x^2}\right)\right) w_x w_y \\ & - \tfrac{1}{\kappa^3}\left(\tfrac{\partial\sigma}{\partial x}\left(\tfrac{\partial\sigma}{\partial y}\tfrac{\partial^2\sigma}{\partial x\partial y} + \tfrac{\partial\sigma}{\partial x}\tfrac{\partial^2\sigma}{\partial x^2}\right) w_x^2 + \tfrac{\partial\sigma}{\partial y}\left(\tfrac{\partial\sigma}{\partial x}\tfrac{\partial^2\sigma}{\partial x\partial y} + \tfrac{\partial\sigma}{\partial y}\tfrac{\partial^2\sigma}{\partial y^2}\right) w_y^2\right) \end{aligned} \tag{3}$$

Plugging in $\tilde{\mathbf{n}}$ for $\mathbf{w}$ in Eq. (3) and using the fact that

$$\nabla\sigma \cdot \mathbf{n} = 0 \ , \tag{4}$$

we get

$$k(\tilde{\mathbf{n}}) = \frac{1}{\kappa}\left(n_x^2\frac{\partial^2\sigma}{\partial x^2} + 2n_x n_y\frac{\partial^2\sigma}{\partial x \partial y} + n_y^2\frac{\partial^2\sigma}{\partial y^2}\right) \tag{5}$$

$$= \frac{1}{\kappa}\Sigma\left(\mathbf{n},\mathbf{n}\right). \tag{6}$$

Now let $\tilde{\mathbf{t}} = \mathbf{u} \times \tilde{\mathbf{n}}$. As above, define

$$k(\tilde{\mathbf{t}}) = \tilde{\mathbf{t}} \cdot \nabla_{\tilde{\mathbf{t}}}\mathbf{u}. \tag{7}$$

If $k(\tilde{\mathbf{t}})$ is expanded out and reduced, some algebra shows that

$$k(\tilde{\mathbf{t}}) = \frac{1}{\kappa^3}\left(t_x^2\frac{\partial^2\sigma}{\partial x^2} + 2t_x t_y\frac{\partial^2\sigma}{\partial x \partial y} + t_y^2\frac{\partial^2\sigma}{\partial y^2}\right) \tag{8}$$

$$= \frac{1}{\kappa^3}\Sigma\left(\mathbf{t},\mathbf{t}\right) \ . \tag{9}$$

□

Notice that $\kappa \geq 1$. Therefore, equality of the two curvature measures holds when $\kappa = 1 \Rightarrow \nabla\sigma = 0$ (i.e. the tangent plane of $\mathcal{G}$ is parallel to the xy-plane).

Theorem 2.2 *A second derivative ridge is always identical to or a subset of a curvature ridge.*

PROOF. We must show that all points along a second-derivative ridge satisfy the conditions of a curvature ridge. Notice that **CR1** is trivially satisfied if **SR1** is true. Hence we must show **CR2**, that is, $k(\tilde{\mathbf{n}})$ is a minimum and less than zero, where $\tilde{\mathbf{n}}$ is the lift of $\mathbf{n}$, and $\mathbf{n}$ satisfies **SR2**, i.e.

$$\Sigma(\mathbf{n},\mathbf{n}) = \min_{\|\mathbf{u}\|=1} \Sigma(\mathbf{u},\mathbf{u}) < 0 \tag{10}$$

with $\mathbf{n}$ orthogonal to $\nabla\sigma$.

From Theorem 2.1, $k(\tilde{\mathbf{n}})$ is necessarily less than zero if Eq. (10) is satisfied. Thus, it is left to show that $k(\tilde{\mathbf{n}})$ is minimized in the (lifted) direction orthogonal to the second-derivative ridge. It should be clear that the scaling introduced in Theorem 2.1 will not affect the difference in ridge definitions for all points in which Σ has a non-negative eigenvalue. Therefore, assume that the eigenvalues of Σ satisfy $\lambda_{\min} < \lambda_{\max} < 0$. Without loss of generality we can assume the second-derivative ridge is locally aligned with the x-axis, i.e. that $\frac{\partial \sigma}{\partial y} = 0$. This, along with Eq. (10), puts Σ in canonical form

$$\Sigma = \begin{bmatrix} \lambda_{\max} & 0 \\ 0 & \lambda_{\min} \end{bmatrix} . \tag{11}$$

Using this relation in Eq. (3) gives

$$k(\hat{\mathbf{u}}) = \frac{1}{\kappa}\left(u_x^2 \lambda_{\max} + u_y^2 \lambda_{\min}\right) - \frac{1}{\kappa^3}\left(\left|\nabla\sigma\right|^2 \lambda_{\max} u_x^2\right) \tag{12}$$

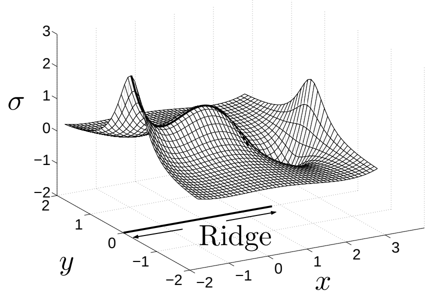
for an arbitrary unit vector $\hat{\mathbf{u}} = (u_x, u_y, 0)$. Notice that both terms in Eq. (12) are positive, hence $k(\hat{\mathbf{u}})$ is minimized if $\hat{\mathbf{u}}$ is in the y-direction (i.e. $\hat{\mathbf{u}} = (0, 1, 0)$), which is the direction orthogonal to the second-derivative ridge. □

3 Example

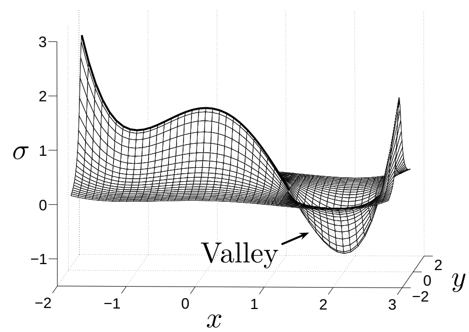
Here we present an example to demonstrate the notions of a curvature ridge and a second derivative ridge. Panels (a) and (b) of Fig. 3 show the graph of an analytical test field σ. It seems intuitive to call the line $y = 0$ a ridge except along the "valley" of the graph, centered around the point $(2, 0)$ in the domain.

It is easily verified that **CR1**, and hence **SR1**, is satisfied for the line $y = 0$. The principal curvatures and second-derivative values given in **CR2** and **SR2** are plotted in Panel (c) of Fig. 3. Panel (d) of Fig. 3 shows a close-up around the value $x = 1.2$. Notice that **SR2** is satisfied for all x less than $x \approx 1.195$ (i.e. up to the 2^{nd}-derivative curvature intersection point shown in Panel (d)) whereas **CR2** is satisfied for all x less than $x \approx 1.2$ (i.e. up to the principal curvature intersection point shown on Panel (d) of Fig. 3). Therefore we see that the second-derivative ridge is a subset of the curvature ridge, which is of course in agreement with Theorem 2.2. In addition, this example shows how the two measures produce near identical results in this case.

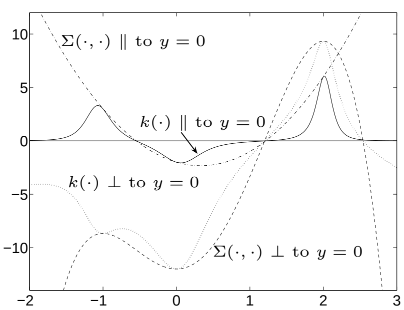
The functional form for σ in this example was chosen to produce an interesting test-case. For actual FTLE fields, σ typically does not vary much along the ridges of the field—in fact, much less than shown in this example. Therefore we can expect the difference between the two measures to be identically zero or non-existent for all practical purposes. For autonomous systems, σ is constant along a ridge, hence the two definitions of ridge are always identical for such systems.



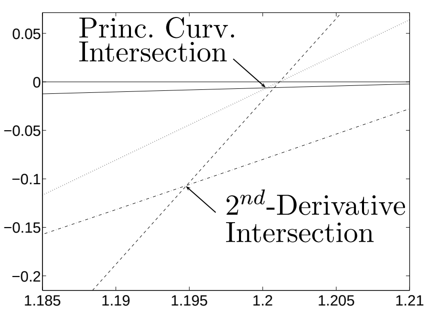
(a) $\sigma = \frac{3x^4 - 4x^3 - 12x^2 + 18}{12(1+4y^2)}$



(b) Side View



(c) Curvature measures evaluated along x-axis (i.e. $y = 0$)



(d) Close Up

Fig. 1. Comparison between ridge definitions. Notice that the second-derivative ridge is slightly shorter than the curvature ridge